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# ADDITION THEOREMS FOR SPHERICAL WAVES

by

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and

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## Abstract

Expansions or "addition theorems" for the spherical wave functions $j_n(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$, $h_n^{(1)}(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$, and $h_n^{(2)}(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$, with reference to the origin O, have been obtained in terms of spherical wave functions with reference to the origin O', where O' has the coordinates $(r_o, \theta_o, \phi_o)$ with respect to O.

Using the above-mentioned addition theorems, we have obtained an expansion for $\dfrac{e^{ikr_{12}}}{r_{12}}$, a wave which is spherically symmetric about the source Q, where Q is referred to the origin O, in terms of products of spherical waves with respect to origin O and spherical waves with respect to O'.

## 1. Introduction

When a plane wave is incident on a configuration of spheres, the scattered field may be obtained by a series of successive approximations. The zero-th order approximation is just the sum of the fields scattered by each individual sphere when the excitation of each sphere is taken to be the original plane wave. Higher order approximations take into account contributions to the excitation of a particular sphere from the waves which the remaining spheres scatter. In particular, the first order approximation is the field scattered by the configuration when the excitation field is taken to be the initial plane wave plus the zero-th order approximation to the scattered wave.

In order to carry out this approximation it is necessary to decompose the scattered field of a sphere with center at O into incoming spherical waves for a second sphere with center at O'. Such a decomposition requires an "addition theorem" which expresses a spherical wave with center at O, in terms of spherical waves with center at O'.

Such an expansion or "addition theorem" for cylindrical waves is well known[1]. Using this expansion, Twersky[2] was able to calculate the scattered field obtained by a plane wave striking a configuration of cylinders, taking into account the contributions to the excitations of a particular element by the radiation scattered by the remaining elements. The analogous case of a plane wave striking a configuration of spheres may be treated with the help of the addition theorems we have obtained in this paper.

The first expansion treated in this paper is that for the standing spherical wave $j_n(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$ with reference to origin O. This spherical wave is represented as an integral of plane waves over all possible directions. These plane waves are also referred to the origin O. A transformation is then made to obtain the plane waves in terms of the origin O'. In order to evaluate the resulting integral it is necessary to have a formula which expresses the product of two associated Legendre functions in terms of a sum of associated Legendre functions. Once this is obtained by using a formula due to Infeld and Hull, the evaluation of the integral is possible and the final form of the expansion is a sum of spherical waves with reference to the origin O'.

The second expansion treated is that for the outgoing spherical wave $h_n^{(1)}(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$ with reference to the origin 0. The same procedure is used as for the expansion of $j_n(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$. However, in this case, the integral is over plane waves with real and complex directions. This leads to convergence difficulties when the transformation from the origin 0 to the origin 0' is effected. These are discussed in Appendix III. The final form of the expansion as a sum of spherical waves with reference to the origin 0' is then obtained.

From the first two expansions we then obtain an expansion for $h_n^{(2)}(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$ with reference to origin 0, in terms of spherical waves referred to origin 0' by using the relationship $h_n^{(1)}(kR) + h_n^{(2)}(kR) = 2j_n(kR)$.

The final problem considered is the expansion of the spherical wave $\frac{e^{ikr_{12}}}{r_{12}}$, where $r_{12}$ is the distance between a point of observation $P(r_1, \theta_1, \phi_1)$ and the source point of the spherical wave $Q(R, \theta, \phi)$, the coordinates of both points being taken with reference to the origin 0. Then, $\frac{e^{ikr_{12}}}{r_{12}}$ can be expressed in terms of a sum of spherical waves with reference to 0. However, using the results of the preceding sections an expansion of $\frac{e^{ikr_{12}}}{r_{12}}$ in terms of products of spherical waves with respect to both origin 0 and to origin 0' is then obtained. This expansion is useful in the variational treatment of diffraction problems.

We wish to express our thanks to Dr. Victor Twersky who suggested this problem.

## 2. The Expansion for $j_n(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$

Consider a point P, which has the spherical coordinates $(R, \theta, \phi)$ with respect to the origin O. A plane wave coming in along the direction $\theta = \alpha$, $\phi = \beta$ can be expressed as follows, in terms of spherical waves with center at O[3]:

$$e^{ikR\cos\gamma} = \sum_{n=0}^{\infty} i^n (2n+1)\, j_n(kR)\, P_n(\cos\gamma),$$

or using the addition theorem for Legendre polynomials,

$$e^{ikR\cos\gamma} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n (2n+1) j_n(kR) \frac{(n-|m|)!}{(n+|m|)!} P_n^{|m|}(\cos\alpha) P_n^{|m|}(\cos\theta) e^{-im(\beta-\phi)}. \quad (1)$$

Here $\gamma$ is the angle between the directions $(\alpha, \beta)$ and $(\theta, \phi)$ so that

$$\cos\gamma = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\phi-\beta).$$

From the expansion (1), an integral representation of elementary spherical wave functions can be obtained. Multiplying both sides of the equation by $P_n^{|m|}(\cos\alpha) e^{im\beta}\sin\alpha$, integrating over $\alpha$ and $\beta$, and using the orthogonality properties of the Legendre functions, we obtain the well-known formula[3]

$$j_n(kR) P_n^{|m|}(\cos\theta) e^{im\phi} = \frac{i^{-n}}{4\pi} \int_0^{2\pi}\int_0^{\pi} e^{ikR\cos\gamma} P_n^{|m|}(\cos\alpha) e^{im\beta} \sin\alpha \, d\alpha \, d\beta. \quad (2)$$

Introduce a new origin O', where O' has coordinates $(r_o, \theta_o, \phi_o)$ with respect to O (Fig. 1), and let $(r, \theta', \phi')$ be the spherical coordinates of the point P with respect to O'.

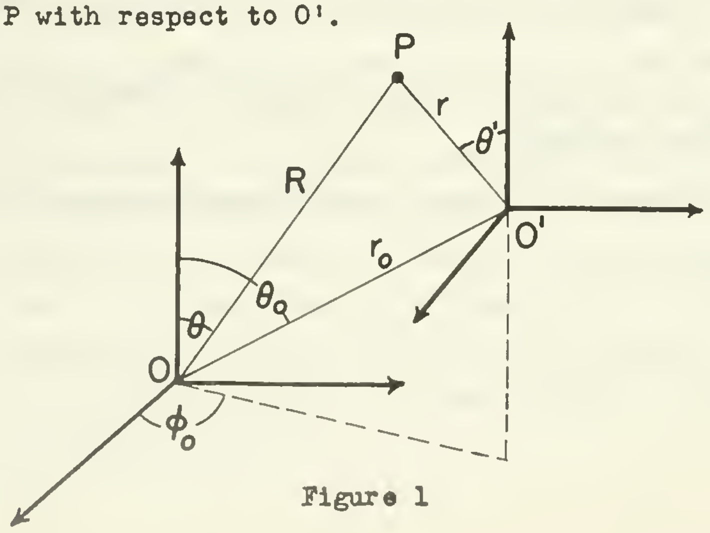

Figure 1

We shall now obtain an expansion for a standing spherical wave around 0 such as $j_n(kR)\, P_n^{|m|}(\cos\Theta)\, e^{im\phi}$ in terms of standing spherical waves around 0' such as $j_\nu(kr)\, P_\nu^{|\mu|}(\cos\Theta')\, e^{i\mu\phi'}$.

First we introduce a set of rectangular coordinates around 0 and a parallel set around 0'. Let (x,y,z) be the coordinates of P relative to 0, and let (x',y',z') be the coordinates of P relative to 0'. We have

$$z = R\cos\Theta \qquad x = R\sin\Theta\cos\phi \qquad y = R\sin\Theta\sin\phi$$
$$z' = r\cos\Theta' \qquad x' = r\sin\Theta'\cos\phi \qquad y' = r\sin\Theta'\sin\phi$$
$$z = z' + z_0 \qquad x = x' + x_0 \qquad y = y' + y_0$$

Now since $\cos\gamma = \cos\Theta\cos\alpha + \sin\Theta\sin\alpha\cos(\beta-\phi)$, it follows that

$$\begin{aligned} R\cos\gamma &= z\cos\alpha + x\sin\alpha\cos\beta + y\sin\alpha\sin\beta \\ &= (z'+z_0)\cos\alpha + (x'+x_0)\sin\alpha\cos\beta + (y'+y_0)\sin\alpha\sin\beta \\ &= r\left[\cos\Theta'\cos\alpha + \sin\Theta'\cos\phi'\sin\alpha\cos\beta + \sin\Theta'\sin\phi'\sin\alpha\sin\beta\right] \\ &\quad + r_0\left[\cos\Theta_0\cos\alpha + \sin\Theta_0\cos\phi_0\sin\alpha\cos\beta + \sin\Theta_0\sin\phi_0\sin\alpha\sin\beta\right], \end{aligned}$$

which can be written as

$$R\cos\gamma = r\cos\gamma' + r_0\cos\gamma_0 . \qquad (3)$$

Here $\gamma'$ is the angle between the direction $(\alpha,\beta)$ and $(\Theta',\phi')$, while $\gamma_0$ is the angle between the direction $(\alpha,\beta)$ and $(\Theta_0,\phi_0)$.

Using equation (3) in equation (2) we have

$$j_n(kR)P_n^{|m|}(\cos\Theta)e^{im\phi} = \frac{i^{-n}}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{ikr\cos\gamma'}e^{ikr_0\cos\gamma_0}P_n^{|m|}(\cos\alpha)\, e^{im\beta}\sin\alpha\, d\alpha\, d\beta. \qquad (4)$$

For the term $e^{ikr\cos\gamma'}$ we use the expansion corresponding to equation (1):

$$e^{ikr\cos\gamma'} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} i^{\nu}(2\nu+1)\,\frac{(\nu-|\mu|)!}{(\nu+|\mu|!}\, j_\nu(kr)P_\nu^{|\mu|}(\cos\Theta')P_\nu^{|\mu|}(\cos\alpha)e^{-i\mu(\phi'-\beta)} .$$

It can be shown (see Appendix I) that the above expansion is a uniformly convergent series and hence when it is substituted into equation (4) the order of summation and integration may be interchanged. Therefore we have

$$j_n(kR)P_n^{|m|}(\cos\theta)e^{in\phi} = \frac{i^{-n}}{4\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\Bigg[i^{\nu}(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\,j_\nu(kr)P_\nu^{|\mu|}(\cos\theta') \qquad (5)$$

$$e^{-i\mu\phi'}\int_0^{2\pi}\int_0^{\pi}\left\{e^{ikr_0\cos\gamma_0}\,P_n^{|m|}(\cos\alpha)P_\nu^{|\mu|}(\cos\alpha)e^{i(m+\mu)\beta}\sin\alpha\right\}d\alpha d\beta\Bigg].$$

In order to simplify the above equation, it is useful to have a formula expressing the product of two associated Legendre functions in terms of a sum of associated Legendre functions. This formula is obtained in the following manner: We first assume that

$$\bar{P}_n^{|m|}(\cos\alpha)\bar{P}_\nu^{|\mu|}(\cos\alpha) = \sum_p \bar{a}_{p,n,\nu}^{|\mu|+|m|}\ \bar{P}_p^{|\mu|+|m|}$$

where $p = \nu+n,\ \nu+n-2,\dots\nu-n$, the $\bar{P}_n^{|m|}(\cos\alpha)$, $\bar{P}_\nu^{|\mu|}(\cos\alpha)$, $\bar{P}_p^{|\mu|+|m|}$ are normalized Legendre functions, and $\cos\alpha$ may take on complex values. Then, multiplying the above expansion by $\bar{P}_q^{|\mu|+|m|}(\cos\alpha)\sin\alpha$ and integrating with respect to $\alpha$, we have, using the orthogonality properties of Legendre functions

$$\int_0^{\pi}\bar{P}_q^{|\mu|+|m|}(\cos\alpha)\bar{P}_n^{|m|}(\cos\alpha)\bar{P}_\nu^{|\mu|}(\cos\alpha)\sin\alpha d\alpha = \bar{a}_q^{|m|+|\mu|}.$$

However from a paper by Infeld and Hull[4] we have the following formula ;

$$\int_0^{\pi}\bar{P}_q^{|\mu|+|m|}(\cos\alpha)\bar{Y}_n^{|m|}(\cos\alpha)\bar{Y}_\nu^{|\mu|}(\cos\alpha)d\alpha = \frac{(n+\nu-q-1)!!\,[(2n+1)(2\nu+1)(2q+1)]^{1/2}}{(n+q-\nu)!!\,(\nu+q-n)!!\,(q+\nu+n+1)!!}$$

$$\left[\frac{(q+|\mu|+|m|)!(q-|\mu|-|m|)!(\nu-|\mu|)!(n-|m|)!}{2(\nu+|\mu|)!(n+|m|)!}\right]^{1/2}$$

$$\times\sum_{i=0}^{q-|\mu|-|m|}\left\{(-1)^{\frac{n-\nu+q}{2}+|m|+i}\ \frac{(n+|m|+i)!(\nu+q-|m|-i)!}{(q-|\mu|-|m|-i)!(i)!(n-|m|-i)!(\nu-q+|m|+i)!}\right\}$$

$$= \bar{a}_q^{|\mu|+|m|}$$

Now, $\bar{P}_q^{|\mu|+|m|} = N_q^{|\mu|+|m|}P_q^{|\mu|+|m|}$, where $N_q^{|\mu|+|m|} = \frac{(2q+1)^{1/2}}{\sqrt{2}}\left[\frac{(q-|m|-|\mu|)!}{(q+|m|+|\mu|)!}\right]^{1/2}$

Using this relationship and similar ones for $\bar{P}_n^{|m|}$ and $\bar{P}_\nu^{|\mu|}$ we arrive at the formula

$$P_n^{|m|}(\cos\alpha)P_\nu^{|\mu|}(\cos\alpha) = \sum_p a_{p,n,\nu}^{|m|+|\mu|}\ P_p^{|m|+|\mu|}(\cos\alpha) \tag{6}$$

where $p = \nu+n,\ \nu+n-2,\ \nu+n-4,\ \cdots\cdots,\ \nu-n$, and where, for any coefficient $a_{q,n,\nu}^{|m|+|\mu|}$, we have the relationship

$$a_{q,n,\nu}^{|\mu|+|m|} = \frac{N_q^{|\mu|+|m|}}{N_n^{|m|}N_\nu^{|\mu|}}\quad \bar{a}_{q,n,\nu}^{|\mu|+|m|}$$

or

$$a_{q,n,\nu}^{|m|+|\mu|} = \frac{(n+\nu-q-1)!!(2q+1)(q-|m|-|\mu|)!}{(n+q-\nu)!!(\nu+q-n)!!(q+\nu+n+1)!!} \times \sum_{i=0}^{q-|m|-|\mu|}\left\{\frac{(-1)^{\frac{n+q-\nu}{2}+|m|+i}(n+|m|+i)!(\nu+q-|m|-i)!}{(q-|m|-|\mu|-i)!(i)!(n-|m|-i)!(\nu-q+|m|+i)!}\right\} \tag{7}$$

Here $(s)!! = s(s-2)(s-4)\ldots$ 2 or 1, $(0)!!=(-1)!!= 1$, and $\nu-n\leq q \leq \nu+n$. (In Appendix II some of the $a_{q,n,\nu}^{|m|+|\mu|}$ have been calculated explicitly for some small values of n and m.)

Using equation (6) in equation (5) we have

$$j_n(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \frac{i^{-n}}{4\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p\left\{\frac{i^\nu(2\nu+1)(\nu-|\mu|)!}{(\nu+|\mu|)!}\,a_{p,n,\nu}^{|\mu|+|m|}\,j_\nu(kr)P_\nu^{|\mu|}(\cos\theta')e^{-i\mu\phi'} \times I_p^{m,\mu}\right\} \tag{8}$$

where $I_p^{m,\mu}$ is an integral of the form

$$I_p^{m,\mu} = \int_0^{2\pi}\int_0^{\pi} e^{ikr_0\cos\gamma_0}\,P_p^{|\mu|+|m|}(\cos\alpha)e^{i(\mu+m)\beta}\,\sin\alpha d\alpha d\beta.$$

It is possible to evaluate these integrals directly[5] (compare equation 4), and we have

$$I_p^{m,\mu} = 4\pi i^p j_p(kr_0)P_p^{|\mu|+|m|}(\cos\theta_0)e^{i(m+\mu)\phi_0}\ . \tag{9}$$

Substituting equation (9) into (8) we have for the expansion of a spherical wave with respect to one origin, in terms of the spherical waves with respect to another origin, the formula

$$j_n(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p\left\{i^{\nu+p-n}\frac{(2\nu+1)(\nu-|\mu|)!}{(\nu+|\mu|)!}\,a_{p,n,\nu}^{|\mu|+|m|}\right.$$

$$\left. j_p(kr_0)j_\nu(kr)P_p^{|\mu|+|m|}(\cos\theta_0)P_\nu^{|\mu|}(\cos\theta')e^{i(m+\mu)\phi_0}\,e^{-i\mu\phi'}\right\}\ . \tag{10}$$

## 3. The Expansion of $h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$ and $h_n^{(2)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$

In order to obtain an addition theorem for the outgoing spherical wave $h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$ we must first get the integral representation for this function. We have the formula[6]

$$\frac{e^{ikr_1}}{ikr_1} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{ikr_1\cos\gamma_1}\sin\alpha\, d\alpha\, d\beta \qquad (11)$$

where $r_1$, $\theta_1$, $\phi_1$ are the coordinates of a point $P_1$ with respect to the origin $O_1$. Now, let the point $P_1$ have spherical coordinates $(R,\theta,\phi)$ with respect to the origin O, and let $O_1$ have coordinates $r^o$, $\phi^o$, $\theta^o$ with respect to O. From equation (3) we see that

$$R\cos\gamma = r^o\cos\gamma^o + r_1\cos\gamma_1$$

or

$$e^{ikr_1\cos\gamma_1} = e^{-ikr^o\cos\gamma^o}\, e^{ikR\cos\gamma} \qquad (12)$$

where as before $\cos\gamma = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\beta-\phi)$.

Now from equation (1) we have

$$e^{-ikr^o\cos\gamma^o} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left\{i^{-n}(2n+1)\frac{(n-|m|)!}{(n+|m|)!}\, j_n(kr^o)P_n^{|m|}(\cos\theta^o)P_n^{|m|}(\cos\alpha)e^{im(\beta-\phi^o)}\right\}. \qquad (13)$$

Therefore, using the formula[7]

$$\frac{e^{ikr_1}}{ikr_1} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-|m|)!}{(n+|m|)!}\, j_n(kr^o)h_n^{(1)}(kR)P_n^{|m|}(\cos\theta^o)P_n^{|m|}(\cos\theta)e^{im(\phi-\phi^o)} \qquad (14)$$

and equation (12) and (13) in equation (11) and comparing the coefficients of each $j_n(kr^o)P_n^{|m|}(\cos\theta^o)$ we find the integral representation to be [8]

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{ikR\cos\gamma}P_n^{|m|}(\cos\alpha)e^{im\beta}\sin\alpha\, d\alpha\, d\beta. \qquad (15)$$

Now let the point P have spherical coordinates $(r',\theta',\phi')$ with reference to another origin O', where O' has coordinates $(r_o,\theta_o,\phi_o)$ with respect to O. (See Fig. 1). From equations (1) and (3) we see that

$$R\cos\gamma = r\cos\gamma' + r_o\cos\gamma_o$$

and that

$$e^{ikr\cos\gamma'} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} i^{\nu}(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\, j_\nu(kr)P_\nu^{|\mu|}(\cos\theta')P_\nu^{|\mu|}(\cos\alpha)e^{-i\mu(\phi'-\beta)}.$$

When these two relations are substituted into equation (15), it can be shown (see Appendix III) that it is possible to exchange the order of summation and integration provided that $r < r_o$. Therefore we have

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \frac{i^{-n}}{2\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\left\{i^{\nu}(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\, j_\nu(kr)P_\nu^{|\mu|}(\cos\theta')e^{-i\mu\phi'}\right.$$

$$\left.\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} e^{ikr_o\cos\theta_o}P_\nu^{|\mu|}(\cos\alpha)P_n^{|m|}(\cos\alpha)e^{i(\mu+m)\beta}\sin\alpha\, d\alpha\, d\beta\right\}. \qquad (16)$$

$$\text{for } r < r_o$$

Now we have the formula (see equation (6))

$$P_\nu^{|\mu|}(\cos\alpha)P_n^{|m|}(\cos\alpha) = \sum_p a_{p,n,\nu}^{|\mu|+|m|} P_p^{|\mu|+|m|}(\cos\alpha),$$

where $p = \nu+n,\ \nu+n-2,\ \cdots\ \nu-n$ and the $a_{p,n,\nu}^{|\mu|+|m|}$ are given by equation (7). Substituting the above expansion into equation (17) we have

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \frac{i^{-n}}{2\pi}\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p \left\{ i^\nu(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\, a_{p,n,\nu}^{|\mu|+|m|}\, j_\nu(kr)P_\nu^{|\mu|}(\cos\theta') \right.$$
$$\left. e^{-i\mu\phi'}K_p^{\mu,m} \right\} \qquad r < r_o, \qquad (18)$$

where the $K_p^{\mu,m}$ are integrals of the form

$$K_p^{\mu,m} = \int_o^{2\pi}\int_o^{\frac{\pi}{2}-i\infty} e^{ikr_o\cos\theta_o} P_p^{|\mu|+|m|}(\cos\alpha)e^{i(\mu+m)\beta}\sin\alpha\, d\alpha\, d\beta.$$

These integrals can be evaluated directly. Using equation (15) we find that

$$K_p^{\mu,m} = 2\pi i^p h_p^{(1)}(kr_o)P_p^{|\mu|+|m|}e^{i(m+\mu)\phi_o}.$$

Therefore, equation (18) becomes

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p \left\{ i^{\nu+p-n}(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\, a_{p,n,\nu}^{|\mu|+|m|}\, j_\nu(kr)h_p^{(1)}(kr_o) \right.$$
$$\left. P_p^{|\mu|+|m|}(\cos\theta_o)P_\nu^{|\mu|}(\cos\theta')e^{i(m+\mu)\phi_o - i\mu\phi'} \right\}, \qquad r < r_o$$

When $r_o < r$, the resulting expansion would be the above equation with $r$ and $r_o$ interchanged. Therefore we have

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p \left\{ i^{\nu+p-n}(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\, a_{p,n,\nu}^{|\mu|+|m|}\, j_\nu(kr_<)h_p^{(1)}(kr_>) \right.$$
$$\left. P_p^{|\mu|+|m|}(\cos\theta_o)P_\nu^{|\mu|}(\cos\theta')e^{i(m+\mu)\phi_o}\, e^{-i\mu\phi'} \right\}, \qquad (19)$$

where $r_<$ is the smaller of $r$ and $r_o$ and $r_>$ is the larger of $r$ and $r_o$. This is the expansion for a spherical wave $h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$ about one origin, in terms of spherical waves around another origin.

To find the expansion for $h_n^{(2)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$ we use the relationship

$$h_n^{(1)}(kR) + h_n^{(2)}(kR) = 2j_n(kR)$$

and the expansions for $j_n(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$ and $h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$. This gives us finally

$$h_n^{(2)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}(2\nu+1)a_{p,n,\nu}^{|\mu|+|m|}j_\nu(kr_<)h_p^{(2)}(kr_>)$$
$$P_p^{|\mu|+|m|}(\cos\theta_o)P_\nu^{|\mu|}(\cos\theta')e^{i(m+\mu)\phi_o}\,e^{-i\mu\phi'}. \qquad (20)$$

## 4. The Expansion of $\frac{e^{ikr_{12}}}{r_{12}}$

Let $P(r_1,\theta_1,\phi_1)$ be a point of observation, and $Q(R,\theta,\phi)$ be the source point of a spherical wave where the coordinates are referred to a fixed coordinate system whose origin is at 0. [Fig. 2]. The distance from Q to P is therefore

$$r_{12} = \sqrt{R^2+r_1^2 - 2Rr_1\cos\gamma}\,,$$

where $\cos\gamma = \cos\theta\,\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)$. Hence we have the expansion[9]

$$\frac{e^{ikr_{12}}}{r_{12}} = \sum_{n=o}^{\infty}(2n+1)P_n(\cos\gamma)j_n(kr_<)h_n^{(1)}(kr_>), \qquad (21)$$

where $r_<$ is the lesser of R and $r_1$, and $r_>$ is the greater of R and $r_1$.

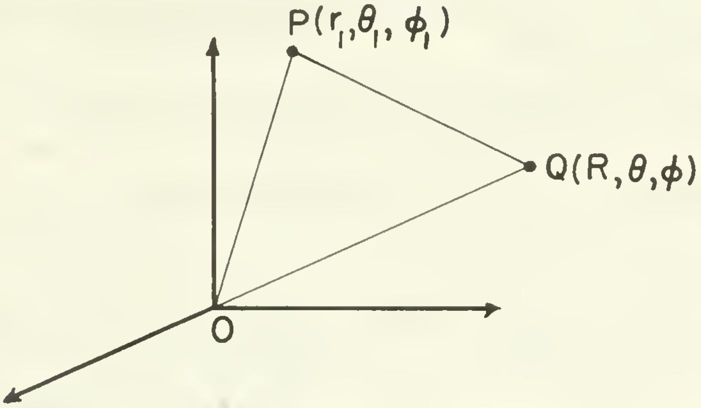

Fig. 2.

Using the following addition formula:

$$P_n(\cos\gamma) = \sum_{m=-n}\frac{(n-|m|)!}{(n+|m|)!}P_n^{|m|}(\cos\theta)P_n^{|m|}(\cos\theta_1)e^{-im(\phi_1-\phi)}$$

in equation (21) we have

$$\frac{e^{ikr_{12}}}{r_{12}} = \sum_{n=o}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-|m|)!}{(n+|m|)!}j_n(kr_<)h_n^{(1)}(kr_>)P_n^{|m|}(\cos\theta)P_n^{|m|}(\cos\theta_1)e^{-im(\phi_1-\phi)}.$$

Consider now the spherical coordinates $(r_2,\theta_2,\phi_2)$ of the point Q with respect to the origin 0', where 0' has coordinates $(r_o,\theta_o,\phi_o)$ with respect to 0. [See Fig. 3]

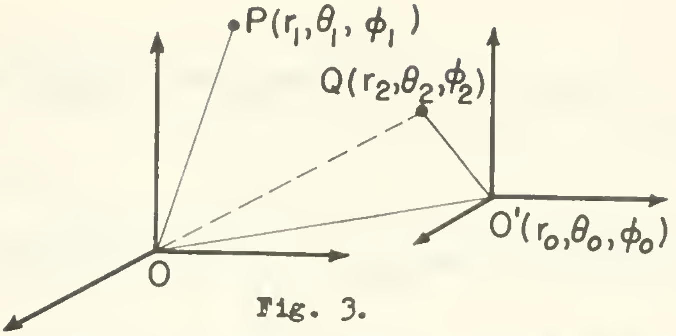

Fig. 3.

Then we can apply to equation (20) the following expansions:

$$j_n(kR)P_n^{|m|}(\cos\theta)e^{im\phi}=\sum_{\nu=o}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}\Bigg\{i^{\nu+p-n}\frac{(2\nu+1)(\nu-|\mu|)!}{(\nu+|\mu|)!}j_p(kr_o)j_\nu(kr_2)a_{p,n,\nu}^{|\mu|+|m|}P_p^{|\mu|+|m|}(\cos\theta_o) \times P_\nu^{|\mu|}(\cos\theta_2)e^{i(m+\mu)\phi_o}e^{-i\mu\phi_2}\Bigg\}, \tag{23}$$

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}=\sum_{\nu=o}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{p}\Bigg\{i^{\nu+p-n}\frac{(2\nu+1)(\nu-|\mu|)!}{(\nu+|\mu|)!}j_\nu(kr'_<)h_p^{(1)}(kr'_>)a_{p,n,\nu}^{|\mu|+|m|} \times P_p^{|\mu|+|m|}(\cos\theta_o)P_\nu^{|\mu|}(\cos\theta_2)e^{i(m+\mu)\phi_o}e^{-i\mu\phi_2}\Bigg\}, \tag{24}$$

where $r'_<$ is the lesser of $r_2$ and $r_o$, and $r'_>$ is the greater of $r_2$ and $r_o$. The final result , after substituting either equation (23) or (24) into (22), is

$$\frac{e^{ikr_{12}}}{r_{12}}=\sum_{n=o}^{\infty}\sum_{\nu=o}^{\infty}\sum_{m=-n}^{n}\sum_{\mu=-\nu}^{\nu}\sum_{p}B_{n,\nu,p}^{|\mu|,|m|}(r_1,r_2;r_o\theta_o,\phi_o)P_n^{|m|}(\cos\theta_1)P_\nu^{|\mu|}(\cos\theta_2)e^{-i(m\phi_1+\mu\phi_2)}, \tag{25}$$

where $p=\nu+n,\nu+n-2,\dots\nu-n$, and where for $r_1>R$

$$B_{n,\nu,p}^{|\mu|,|m|}(r_1,r_2;r_o\theta_o,\phi_o)=i^{\nu+p-n}\frac{(2n+1)(2\nu+1)(n-|m|)!(\nu-|\mu|)!}{(n+|m|)!(\nu+|\mu|)!}a_{p,n,\nu}^{|\mu|+|m|}j_p(kr_o)h_n^{(1)}(kr_1) \times j_\nu(kr_2)P_p^{|\mu|+|m|}(\cos\theta_o)e^{i(m+\mu)\phi_o},$$

and for $r_1<R$

$$B_{n,\nu,p}^{|\mu|,|m|}(r_1,r_2;r_o\theta_o,\phi_o)=i^{\nu+p-n}\frac{(2n+1)(2\nu+1)(n-|m|)!(\nu-|\mu|)!}{(n+|m|)!(\nu+|\mu|)!}a_{p,n,\nu}^{|\mu|+|m|}h_p^{(1)}(kr'_>)j_n(kr_1) \times j_\nu(kr'_<)P_p^{|\mu|+|m|}(\cos\theta_o)e^{i(m+\mu)\phi_o}.$$

If we let

$$A_{n,\nu,p}^{|\mu|,|m|}(r_2;\theta_o,\phi_o)=i^{\nu+p-n}\frac{(2n+1)(n-|m|)!}{(n+|m|)!}\frac{(2\nu+1)(\nu-|\mu|)!}{(\nu+|\mu|)!}a_{p,n,\nu}^{|\mu|+|m|}P_p^{|\mu|+|m|}(\cos\theta_o)e^{i(m+\mu)\phi_o}$$

then for $R>r_1$

$$B^{|\mu|,|m|}_{p,\nu,n}(r_1,r_2;r_0,\theta_0,\phi_0)=j_\nu(kr'_<)h^{(1)}_p(kr'_>)j_n(kr_1)A^{|\mu|,|m|}_{n,\nu,p}(r_2;\theta_0,\phi_0),$$

and for $R < r_1$

$$B^{|\mu|,|m|}_{p,\nu,n}(r_1,r_2;r_0,\theta_0,\phi_0)=j_\nu(kr_2)j_p(kr_0)h^{(1)}_n(kr_1)A^{|\mu|,|m|}_{n,\nu,p}(r_2;\theta_0,\phi_0) .$$

## Appendix I.

To show that the expansion

$$e^{ikr\cos\gamma'} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} i^{\nu}(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\, j_{\nu}(kr)P_{\nu}^{|\mu|}(\cos\theta')P_{\nu}^{|\mu|}(\cos\alpha)e^{-i\mu(\phi'-\beta)}$$

is a uniformly convergent series, we first write it in the form

$$e^{ikr\cos\gamma'} = \sum_{\nu=0}^{\infty} i^{\nu}(2\nu+1)j_{\nu}(kr)P_{\nu}(\cos\gamma').$$

Now we know that for $0 \leq \gamma' \leq 2\pi$, $|P_{\nu}(\cos\gamma')| \leq 1$, and also $|e^{-i\mu(\phi'-\beta)}| = 1$. For large values of $\nu$, $|j_{\nu}(kr)|$ behaves like $\frac{(\frac{kr}{2})^{\nu}}{\nu!}$. Hence we have

$$\sum_{\nu=0}^{\infty} \left| i^{\nu}(2\nu+1)j_{\nu}(kr)P_{\nu}(\cos\gamma')\right| \leq \sum_{\nu=0}^{\infty} (2\nu+1)\frac{(\frac{kr}{2})^{\nu}}{\nu!}.$$

However, $\sum_{\nu=0}^{\infty} (2\nu+1)\frac{(\frac{kr}{2})^{\nu}}{\nu!}$ is a convergent series, and therefore it follows that

$$\sum_{\nu=0}^{\infty} i^{\nu}(2\nu+1)j_{\nu}(kr)P_{\nu}(\cos\gamma')$$

is a uniformly convergent series.

## Appendix II.

For general values of $\nu$, n, $|\mu|$ and $|m|$, a formula has been obtained (see equation 7) for the coefficients $a_{p,n,\nu}^{|\mu|+|m|}$. Using this formula, we have calculated the coefficients for some special values of n and m. It is sufficient to evaluate the $a_{p,n,\nu}^{|\mu|+|m|}$ for positive values of m only, since from these the coefficients for negative values of m are readily obtainable and depend upon the definition of $P_n^{-m}(x)$ in terms of $P_n^m(x)$.

We see that for n=1 the $a_{p,1,\nu}^{|\mu|+|m|} = 0$ unless $p = \nu+1$ or $\nu-1$ :

| n=1 | $a_{\nu+1,1,\nu}^{|\mu|+|m|}$ | $a_{\nu-1,1,\nu}^{|\mu|+|m|}$ |
|---|---|---|
| m=0 | $\frac{(\nu-|\mu|+1)}{(2\nu+1)}$ | $\frac{(\nu+|\mu|)}{2\nu+1}$ |
| m=1 | $\frac{1}{2\nu+1}$ | $\frac{-1}{2\nu+1}$ . |

For n=2, the $a^{|\mu|+|m|}_{p,2,\nu} = 0$ unless $p=\nu+2, \nu, \nu-2$ :

| n=2 | $a^{\vert\mu\vert+\vert m\vert}_{\nu+2,2,\nu}$ | $a^{\vert\mu\vert+\vert m\vert}_{\nu,2,\nu}$ | $a^{\vert\mu\vert+\vert m\vert}_{\nu-2,2,\nu}$ |
|---|---|---|---|
| m=0 | $\frac{3}{2}\frac{(\nu-\vert\mu\vert+2)(\nu-\vert\mu\vert+1)}{(2\nu+3)(2\nu+1)}$ | $\frac{(\nu^2+\nu-3\vert\mu\vert^2)}{(2\nu+3)(2\nu-1)}$ | $\frac{3}{2}\frac{(\nu+\vert\mu\vert)(\nu+\vert\mu\vert-1)}{(2\nu+1)(2\nu-1)}$ |
| m=1 | $3\frac{(\nu-\vert\mu\vert+1)}{(2\nu+3)(2\nu+1)}$ | $\frac{3}{2}\frac{(\nu+\vert\mu\vert)}{(2\nu+3)(2\nu-1)}$ | $-3\frac{(\nu+\vert\mu\vert)}{(2\nu+1)(2\nu-1)}$ |
| m=2 | $\frac{3}{(2\nu+3)(2\nu+1)}$ | $\frac{-6}{(2\nu+3)(2\nu-1)}$ | $\frac{3}{(2\nu+1)(2\nu-1)}$ . |

When n=3, $a^{|\mu|+|m|}_{p,3,\nu} = 0$ unless $p = \nu+3,\ \nu+1,\ \nu-1,\ \nu-3.$

For m=0

$$a^{|\mu|+|m|}_{\nu+3,3,\nu} = \frac{5}{2}\ \frac{(\nu-|\mu|+3)(\nu-|\mu|+2)(\nu-|\mu|+1)}{(2\nu+5)(2\nu+3)(2\nu+1)}$$

$$a^{|\mu|+|m|}_{\nu+1,3,\nu} = \frac{3}{2}\ \frac{(\nu-|\mu|+1)(\nu^2+2\nu-5|\mu|^2)}{(2\nu+5)(2\nu+1)(2\nu-1)}$$

$$a^{|\mu|+|m|}_{\nu-1,3,\nu} = \frac{3}{2}\ \frac{(\nu+|\mu|)(\nu^2-5|\mu|^2-1)}{(2\nu+3)(2\nu+1)(2\nu-3)}$$

$$a^{|\mu|+|m|}_{\nu-3,3,\nu} = \frac{5}{2}\ \frac{(\nu+|\mu|)(\nu+|\mu|-1)(\nu+|\mu|-2)}{(2\nu+1)(2\nu-1)(2\nu-3)}\ .$$

Appendix III.

In order to prove the validity of exchanging the order of summation and integration in equation (16), we first write it in the form

$$h^{(1)}_n(kR)P^{|m|}_n(\cos\theta)e^{im\phi} = \frac{i^{-n}}{2\pi}\lim_{c\to\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{2}-ic}\left\{\left[\sum_{\nu=0}^{\infty} i^{\nu}(2\nu+1)j_\nu(kR)P_\nu(\cos\gamma')\right]\right.$$

$$\left. e^{ikr_0\cos\gamma_0}\ P^{|m|}_n(\cos\alpha)e^{im\beta}\sin\alpha d\alpha d\beta\right\}. \qquad (26)$$

However, $|j_\nu(kr)| \le \frac{(\frac{kr}{2})^\nu}{\nu!}$ and $|P_\nu(\cos\gamma')| \le |\cos\gamma' + \sqrt{\cos^2\gamma'-1}|^\nu$ ;

hence

$$\sum_{\nu=0}^{\infty} |i^\nu(2\nu+1)j_\nu(kR)P_\nu(\cos\gamma')| \le \sum_{\nu=0}^{\infty} \frac{(2\nu+1)|\frac{kr}{2}(\cos\gamma'+\sqrt{\cos^2\gamma'-1}|^\nu}{\nu!},$$

where the right hand side is a convergent series, and therefore

$$\sum_{\nu=0}^{\infty} i^\nu(2\nu+1)j_\nu(kr)P_\nu(\cos\gamma')$$

is a uniformly convergent series for any finite values of r and $\gamma'$.

Now, in the finite region containing the segment of the contour of integration included between and 0 and $\frac{\pi}{2} - iC$, $e^{ikr_0\cos\gamma_0}P_n^{|m|}(\cos\alpha)\sin\alpha e^{im\beta}$ is bounded, and hence the series

$$e^{ikr_0\cos\gamma_0}P_n^{|m|}(\cos\alpha)\sin\alpha e^{im\beta}\sum i^\nu(2\nu+1)j_\nu(kr)P_\nu(\cos\gamma')$$

is uniformly convergent in this region. Therefore we can write equation (26) as

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \frac{i^{-n}}{2\pi}\lim_{C\to\infty}\left\{\sum_{\nu=0}^{\infty} i^\nu(2\nu+1)j_\nu(kr)\int_0^{2\pi}\int_0^{\frac{\pi}{2}-iC} P_\nu(\cos\gamma')P_n^{|m|}(\cos\alpha)\right.$$

$$\left. e^{ikr_0\cos\gamma_0}e^{im\beta}\sin\alpha d\alpha d\beta\right\}. \tag{27}$$

Equation (27) can then be put in the form

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \frac{i^{-n}}{2\pi}\sum_{\nu=0}^{\infty} i^\nu(2\nu+1)j_\nu(kr)\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} P_\nu(\cos\gamma')P_n^{|m|}(\cos\alpha)$$

$$e^{ikr_0\cos\gamma_0}e^{im\beta}\sin\alpha d\alpha d\beta$$

$$-\frac{i^{-n}}{2\pi}\lim_{C\to\infty}\left\{\sum_{\nu=0}^{\infty} i^\nu(2\nu+1)j_\nu(kr)\int_0^{2\pi}\int_{\frac{\pi}{2}-iC}^{\frac{\pi}{2}-i\infty} P_\nu(\cos\gamma')P_n^{|m|}(\cos\alpha)e^{ikr_0\cos\gamma_0}e^{im\beta}\sin\alpha d\alpha d\beta\right\}. \tag{28}$$

If we can now show that

$$K = \frac{i^{-n}}{2\pi}\lim_{C\to\infty}\left\{\sum_{\nu=0}^{\infty} i^\nu(2\nu+1)j_\nu(kr)\int_0^{2\pi}\int_{\frac{\pi}{2}-iC}^{\frac{\pi}{2}-i\infty} P_\nu(\cos\gamma')P_n^{|m|}(\cos\alpha)e^{ikr_0\cos\gamma_0}e^{im\beta}\sin\alpha d\alpha d\beta\right\} = 0, \tag{29}$$

then the proof will be completed.

To prove that $K \to 0$ as $C \to \infty$, we first substitute into equation (29) the expansion

$$P_\nu(\cos\gamma') = \sum_{\mu=-\nu}^{\nu} \frac{(\nu-|\mu|)!}{(\nu+|\mu|)!} \cdot P_\nu^{|\mu|}(\cos\theta')P_\nu^{|\mu|}(\cos\alpha)e^{-i\mu(\phi'-\beta)}.$$

(Since this is a finite sum, we may interchange the order of summation and integration.) This gives us

$$K = \frac{i^{-n}}{2\pi}\lim_{C\to\infty}\left\{\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} i^\nu(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\, j_\nu(kr)P_\nu^{|\mu|}(\cos\theta')e^{-i\mu\phi'}\int_0^{2\pi}\int_{\frac{\pi}{2}-iC}^{\frac{\pi}{2}-i\infty} P_\nu^{|\mu|}(\cos\alpha)P_n^{|m|}(\cos\alpha)\right.$$
$$\left. e^{ikr_o\cos\gamma_o}e^{i(m+\mu)\beta}\sin\alpha\, d\alpha\, d\beta\right\}. \qquad (30)$$

Now, in the $(\alpha,\beta)$ plane we perform a rotation such that

$$\cos\alpha' = \cos\gamma_o = \cos\alpha\cos\theta_o + \sin\alpha\sin\theta_o\cos(\beta-\phi_o)$$
$$\beta' = \beta-\phi_o .$$

Then $\cos\alpha = \cos\alpha'\cos\theta_o + \sin\alpha'\sin\theta_o\cos(\beta'+\phi_o)$ and the limits of integration remain the same. Hence K becomes

$$K = \frac{i^{-n}}{2\pi}\lim_{C\to\infty}\left\{\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} i^\nu(2\nu+1)\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}\; j_\nu(kr)e^{-i\mu\phi'}\; P_\nu^{|\mu|}(\cos\theta')\right.$$

$$\int_0^{2\pi}\int_{\frac{\pi}{2}-iC}^{\frac{\pi}{2}-i\infty}\left[e^{ikr_o\cos\alpha'} P_\nu^{|\mu|}(\cos\alpha'\cos\theta_o+\sin\alpha'\sin\theta_o\cos(\beta'+\phi_o)) \qquad (31)\right.$$
$$\left.\left. P_n^{|m|}(\cos\alpha'\cos\theta_o+\sin\alpha'\sin\theta_o\cos(\beta'+\phi_o))e^{i(m+\mu)(\beta'-\phi_o)}\sin\alpha'\,d\alpha'\,d\beta'\right]\right\}$$

Now for large values of $|z|$, we have the following asymptotic form for $P_p^{|q|}(z)$,[10]

$$P_p^{|q|}(z) \simeq \frac{2^p\Gamma(p+\frac{1}{2})(z)^p}{(p-|q|)!\,\Gamma(\frac{1}{2})} .$$

If we let $\alpha' = \frac{\pi}{2} - i\psi$, where $C \le \psi \le \infty$, then for very large $\psi$, $\cos\alpha'$ behaves like $\frac{ie^\psi}{2}$ and $\sin\alpha'$ behaves like $\frac{e^\psi}{2}$. Hence, using the above asymptotic form we find that

$$\left| P_\nu^{|\mu|}\left(\cos\alpha'\cos\theta_o+\sin\alpha'\sin\theta_o\cos(\beta'+\phi_o)\right)\right| \simeq \left|\frac{2^\nu\Gamma(\nu+\frac{1}{2})(\frac{e^\psi}{2})^\nu}{(\nu-|\mu|)!\,\Gamma(\frac{1}{2})}\right|$$
$$\left| P_n^{|m|}\left(\cos\alpha'\cos\theta_o+\sin\alpha'\sin\theta_o\cos(\beta'+\theta_o)\right)\right| \simeq \left|\frac{2^n\Gamma(n+\frac{1}{2})(\frac{e^\psi}{2})^m}{(n-|m|)!\,\Gamma(\frac{1}{2})}\right| .$$

Therefore, on substituting the above expressions into equation (29), and then taking the absolute value of K and integrating over $\beta'$, we have

$$|K| \leq \left| \frac{i^{-n}\Gamma(n+\frac{1}{2})}{2\pi\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(n-|m|)!} \right| \lim_{C \to \infty} \left\{ \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left| \frac{i^{\nu}(2\nu+1)\Gamma(\nu+\frac{1}{2})}{(\nu+|\mu|)!} j_{\nu}(kr)P_{\nu}^{|\mu|}(\cos\theta')e^{-i\mu\phi'} \right| \right.$$
$$\left. \times\, \pi \int_{C}^{\infty} (e^{\Psi})^{\nu+n} e^{\frac{-kr_0}{2} e^{\Psi}} e^{\Psi} d\Psi \right\}. \qquad (32)$$

The integral in equation (30) is now a portion of the real axis. Letting $t = e^{\Psi}$ the integral becomes

$$\int_{C}^{\infty} (e^{\Psi})^{\nu+n} e^{\frac{-kr_0}{2} e^{\Psi}} e^{\Psi} d\psi = \int_{D}^{\infty} t^{\nu+n} e^{\frac{-kr_0}{2} t} dt \qquad (33)$$

where $D = e^{C}$ and therefore as $C \to \infty$, $D \to \infty$. Integrating equation (33) by parts, it becomes

$$\int_{D}^{\infty} t^{\nu+n} e^{\frac{kr_0}{2} t} dt = \left\{ - \frac{2t^{\nu+n} e^{-\frac{kr_0}{2} t}}{kr_0} \Bigg|_{D}^{\infty} + \frac{2(\nu+n)}{kr_0} \int_{D}^{\infty} t^{\nu+n-1} e^{-\frac{kr_0}{2} t} dt \right\}. \qquad (34)$$

Therefore

$$\int_{C}^{\infty} (e^{\Psi})^{\nu+n} e^{-\frac{kr_0}{2} e^{\Psi}} e^{\Psi} d\psi \leq \frac{2D^{\nu+n} e^{\frac{-kr_0 D}{2}}}{kr_0}, \qquad (35)$$

and using thds inequality with equation (32), the result is

$$|K| \leq \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})(n-|m|)!kr_0} \lim_{D \to \infty} D^{n} e^{\frac{-kr_0 D}{2}} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{2(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{(\nu+|\mu|)!} |j_{\nu}(kr)| |P_{\nu}^{|\mu|}(\cos\theta')| D^{\nu}$$

However, $|P_{\nu}^{|\mu|}(\cos\theta')| \leq \frac{2}{\sqrt{\pi}\sqrt{2\sin\theta'}} \quad \frac{(\nu+|\mu|)!}{\Gamma(\nu+\frac{3}{2})}$

and $\quad |j_{\nu}(kr)| \simeq \frac{(\frac{kr}{2})^{\nu}}{\nu!}$ for large values of $\nu$.

Putting these values for $|P_{\nu}^{|\mu|}(\cos\theta')|$ and $|j_{\nu}(kr)|$ into equation (34) and writing $(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})$ as $\Gamma(\nu+\frac{3}{2})$, we have

$$|K| \leq \frac{\Gamma(n+\frac{1}{2})}{\pi^{3/2} kr_0 \sqrt{2\sin\theta'}(n-|m|)!} \lim_{D \to \infty} D^{n} e^{\frac{-kr_0}{2} D} \sum_{\nu=0}^{\infty} \frac{(\frac{krD}{2})^{\nu}}{\nu!}. \qquad (37)$$

However, $\sum_{\nu=0}^{\infty} \frac{\left(\frac{krD}{2}\right)^{\nu}}{\nu!} = e^{\frac{krD}{2}}$, and therefore equation (37) becomes

$$|K| \leq A \lim_{D\to\infty} D^{n} e^{\frac{k}{2}(r-r_0)D} \qquad (38)$$

where A is the constant $\dfrac{\Gamma(n+\frac{1}{2})}{\pi^{3/2} kr_0\sqrt{2\sin\theta'}\,(n-|m|)!}$

If we now let $D \to \infty$ we see that $|K| \to 0$ provided that $r < r_0$. Therefore equation (28) becomes

$$h_n^{(1)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi} = \frac{i^{-n}}{2\pi}\sum_{\nu=0}^{\infty} i^{\nu}(2\nu+1)j_{\nu}(kr)\int_0^{2\pi}\int_0^{\frac{\pi}{2}-i\infty} P(\cos\gamma')P_n^{|m|}(\cos\alpha)$$

$$e^{ikr_0\cos\gamma_0}e^{im\beta}\sin\alpha\, d\alpha\, d\beta \qquad \text{when } r < r_0 \qquad (39)$$

and when $P_{\nu}(\cos\gamma')$ is expanded in terms of $P_{\nu}^{|\mu|}(\cos\alpha)P_{\nu}^{|\mu|}(\cos\theta')$ we have equation (16).

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[6] Stratton, J.A.; Electromagnetic Theory, p.578

[7] Ibid. p.414

[8] A similar integral representation for $h_n^{(2)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$ has been obtained by Suto, Yasuo, Bull.Earthquake Res. Institute, Tokyo, 28 (1950) 1-22 and 175-217. In this paper Suto obtains an addition theorem for $h_n^{(2)}(kR)P_n^{|m|}(\cos\theta)e^{im\phi}$ also; however, his expansion is valid only for a translation of the origin along the z axis. The addition theorem he obtains contains powers of kr instead of the Bessel functions $j_\nu(kr)$, and the coefficients of his expansion are not given by a general formula but must be evaluated by a recurrence relation.

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